Mixed-Integer Optimization for the Combined capacitated Facility Location-Routing Problem

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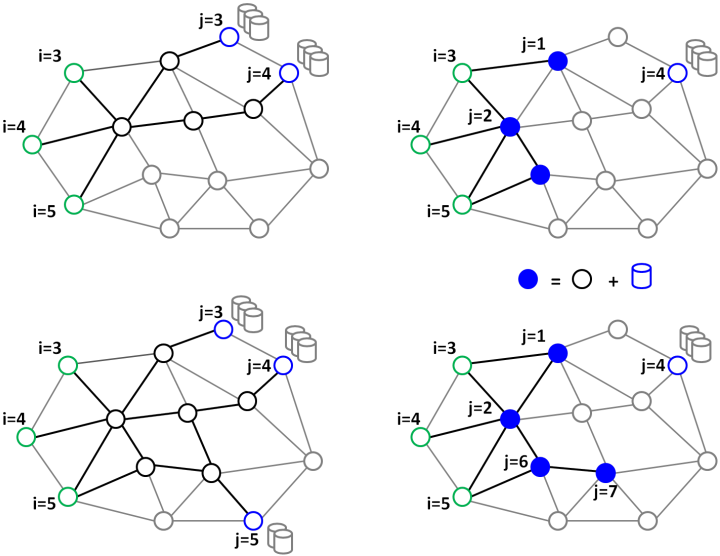
Overall Picture



Model (1)

capacitated Facility Location Problem (cFLP)

- Graph $G = (\mathcal{V}, \mathcal{E})$ where vertex set $\mathcal{V}$ represents
  - Demand originating points $\mathcal{I} \subseteq \mathcal{V}$
  - Set of potential facility locations (sites) $\mathcal{J} \subseteq \mathcal{V}$
- $\forall j \in \mathcal{J}$ of finite capacity b_j
  - Facility opening cost φ_j
  - Assignment cost ϖ_{ij} (allocation of demand a_i from customer demand point i)
- Choose subset of potential locations where to install a facility and assign every client i with known demand a_i to single or to (sub)set of open facilities without exceeding their capacity b_j

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Goal

Find set of facilities to open (location) and assign demands to open facilities (allocation) that minimize the sum of

- Opening/installation cost of selected facilities
- Customer demand supplying cost at each facility
- Cost of connecting each customer demand to subset of selected facilities

Model (2)

Properties

1. **Hard-capacitated**: only one facility may be installed at each location $j \in \mathcal{J}$ with finite capacity b_j
2. **Multi-source**: each client i may be served by multiple sources (facilities $j \in \mathcal{J}$)
3. **Multi-product**: each opened facility j offers multiple (k) commodities a.k.a products (e.g., digital/content objects of different types)

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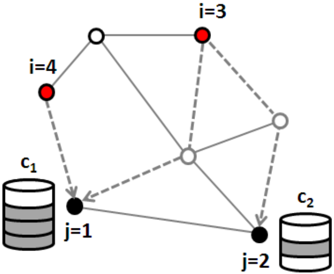

Model (3)

Properties

4. **Symmetric connection/routing cost**: optimal solution to client-to-server problem $\equiv$ optimal solution to server-to-client problem
5. **Shared-capacity model**:
   - Installed capacity shared among objects hosted by each facility
   - Difference compared to physical goods: **single copy** of each object hosted at installed facilities even if assigned to multiple customer demands a_i

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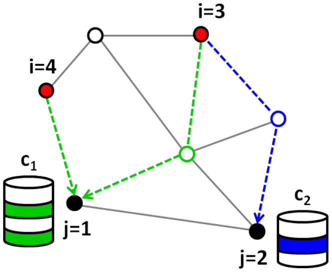

Facility Location-Routing Problem

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  $\Rightarrow$ Allocation (transportation, routing) cost not limited to graph distance
- When routing topology determined endogenously, more effective to change routing decisions instead of locating new facilities

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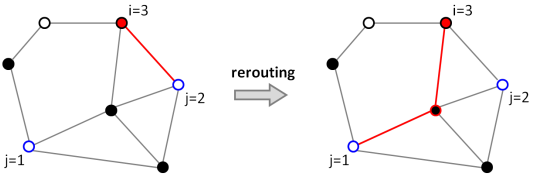

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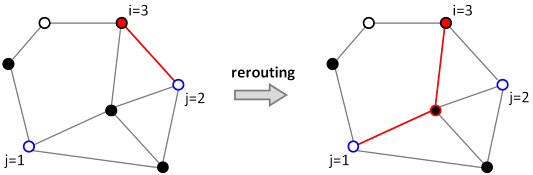
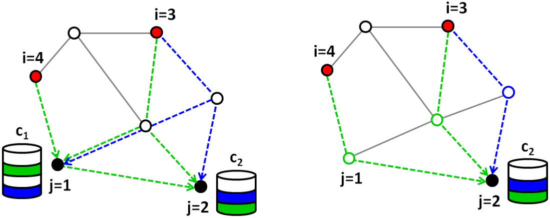


Main idea

- **Combination** of multi-source multi-product capacitated facility location problem (MSMP-cFLP) **for digital goods** with flow routing problem: MSMP-cFLFRP
- Modeled and solved **independently** $\rightarrow$ Modeled and solved **simultaneously**

Reliable Facility Location vs. MSMP-cFLFRP

- Demands protection (Snyder2005): RFLP (and capacitated RFLP)



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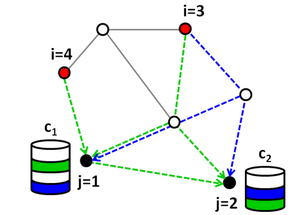
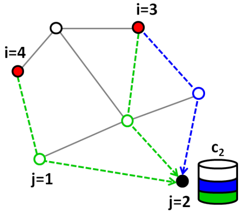
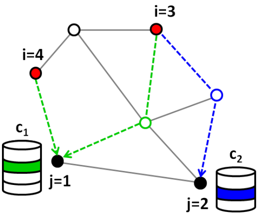
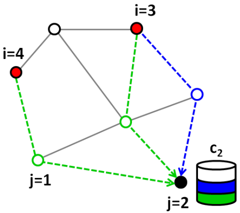




- Demands rerouting (this paper): MSMP-cFLFRP





Input: Data and Parameters

Data

- Finite graph $G = (\mathcal{V}, \mathcal{E})$ with edge set $\mathcal{E}$ and vertex set $\mathcal{V}$
  - Set of demand originating points $\mathcal{I} \subseteq \mathcal{V}$
  - Set of potential facility locations $\mathcal{J} \subseteq \mathcal{V}$
- Set $\mathcal{K}$ ($|\mathcal{K}| = K$): family of products (commodities) that can be hosted by each facility $j \in \mathcal{J}$ ($|\mathcal{J}| = J$)
- Demand set $\mathcal{A}$
  - a_{ik}: size of requested product of type $k \in \mathcal{K}$ initiated by customer demand point $i \in \mathcal{I} \subseteq \mathcal{V}$
  - Total demand over all product types $k \in \mathcal{K}$: $A = \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik}$

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Parameters

- b_j: capacity of facility opened at location $j \in \mathcal{J}$ (storage capacity)
- q_{uv}: nominal capacity of arc (u, v) from node u to v

Variables and Costs

Variables

- Real variable x_{ijk}: fraction of demand a_{ik} requested by customer demand node i for product type k satisfied/served by facility j (opened/installed at $u \in \mathcal{V}$)
- Binary variable $y_j = 1$ if facility j of capacity b_j opened/installed at node $u \in \mathcal{V}$ (0 otherwise)
- Binary variable $z_{jk} = 1$ if product type k provided at (opened) facility j (0 otherwise)
- Continuous flow variable $f_{uv,ijk}$: amount of traffic flowing on arc (u, v) in supply of customer demand i for product k assigned to opened facility j

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Cost

- φ_j: cost of opening/installing a facility at site j
- κ_{ijk}: cost of assigning to facility opened at site j the fraction of demand a_{ik} issued by customer demand point i for product k
- τ_{uv}: cost of routing one unit of traffic along arc (u, v)

MIP formulation

Objective function

1. Facility location cost: $\sum_{j \in \mathcal{J}} \varphi_j y_j$
2. Demand allocation cost: $\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \kappa_{ijk} x_{ijk}$
3. Traffic routing cost: $\sum_{(u,v) \in \mathcal{E}} \tau_{uv} \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} f_{uv,ijk}$

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MIP formulation

$$\min \sum_{(u,v)\in\mathcal{E}} \tau_{uv} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} f_{uv,ijk} + \sum_{j\in\mathcal{V}} \varphi_j y_j + \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \kappa_{ijk} x_{ijk} \qquad (1)$$

MSP-cFLP Constraints (1)

- **Demand satisfaction constraints**: demand a_{ik} for product type k issued by each customer i shall be satisfied:

$$\sum_{j \in \mathcal{J}} x_{ijk} = 1, i \in \mathcal{I}, k \in \mathcal{K}, a_{ik} > 0 \tag{2}$$

- **Product availability**: product type k available on facility j only if j opened
Forbids assigning products to closed facilities:

$$z_{jk} \leq y_j, j \in \mathcal{J}, k \in \mathcal{K} \tag{3}$$

- Demand fraction x_{ijk} satisfiable by facility j only if product k available at j
Forbids delivery from facility j of product type k to demand node i if product type k unavailable at facility j

$$x_{ijk} \leq z_{jk}, i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K} \tag{4}$$

MSMP-cFLP Constraints (2)

Facility capacity constraints:

- For physical goods (canonical cFLP):

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} x_{ijk} \leq b_j y_j, \forall j \in \mathcal{J} \tag{5}$$

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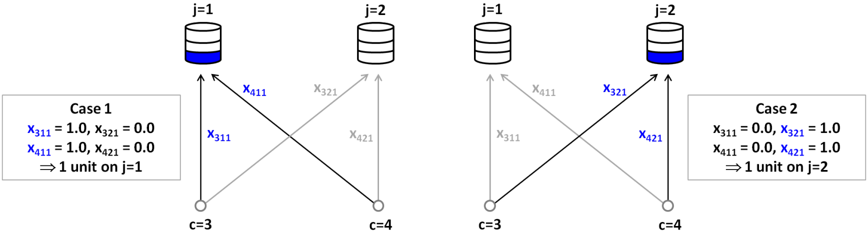
- For digital goods:
  - Sum of fractions x_{ijk} assigned to opened facility $j \in \mathcal{J}$ does not exceed its max. capacity b_j
  - Set of d identical demands (same product type k of size s) assigned to j consumes s units of facility capacity at j instead of $d.s$ units

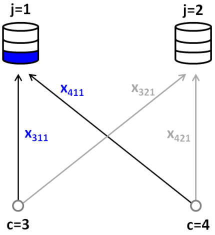
$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} \frac{x_{ijk}}{\sum_{\ell \in \mathcal{L}} x_{\ell jk}} \leq b_j y_j, \forall j \in \mathcal{J} \qquad (6)$$

where, $\mathcal{L}(\subseteq \mathcal{I}) \triangleq$ set of identical demands assigned to the same facility j

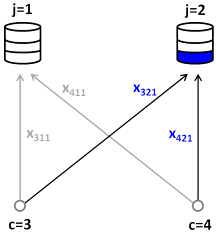
Example


Case 1
x311 = 1.0, x321 = 0.0
x411 = 1.0, x421 = 0.0
⇒ 1 unit on j=1
j=1
j=2
x411
x321
x311
x421
c=3
c=4
j=1
j=2
x411
x321
x311
x421
c=3
c=4
Case 2
x311 = 0.0, x321 = 1.0
x411 = 0.0, x421 = 1.0
⇒ 1 unit on j=2

Example

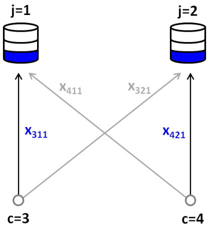


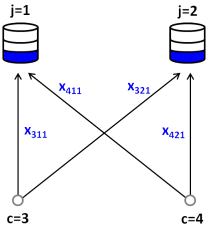
Case 1
$x_{311} = 1.0$, $x_{321} = 0.0$
$x_{411} = 1.0$, $x_{421} = 0.0$
$\Rightarrow$ 1 unit on j=1



Case 2
$x_{311} = 0.0$, $x_{321} = 1.0$
$x_{411} = 0.0$, $x_{421} = 1.0$
$\Rightarrow$ 1 unit on j=2

Case 3
$x_{311} = 1$
$x_{321} = 0$
$x_{411} = 0$
$x_{421} = 1$
$\Rightarrow$ 1 unit on j=1 and j=2





Case 4
$x_{311} = 0 < a < 1$
$x_{321} = 0 < 1 - a < 1$
$x_{411} = 0 < 1 - b < 1$
$x_{421} = 0 < b < 1$
$\Rightarrow$ 1 unit on j=1 and j=2

Constraints linking MSMP-cFLP & Flow Routing Problem

- **Individual flow constraints** on arc (u, v): traffic flow associated to customer i demand for product type k (a_{ik}) directed to facility j along arc (u, v)

$$f_{uv,ijk} \leq \min(q_{uv}, a_{ik}x_{ijk}), (u, v) \in \mathcal{E}, i \in \mathcal{I}, j \in \mathcal{J}, k \in \mathcal{K} \quad (7)$$

- **Aggregated flow constraints** on arc (u, v): load (sum of traffic flows) on individual arcs $(u, v) \in \mathcal{E}$ does not exceed their nominal capacity q_{uv}

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} f_{uv,ijk} \leq q_{uv}, (u, v) \in \mathcal{E} \quad (8)$$

- **Flow conservation constraints**:

$$a_{ik}x_{iik} + \sum_{v:(i,v) \in \mathcal{E}} \sum_{j \in \mathcal{J}} f_{iv,ijk} = a_{ik}, i \in \mathcal{I}, k \in \mathcal{K}, i \neq j, a_{ik} > 0 \quad (9)$$

$$\sum_{v:(v,u) \in \mathcal{E}} \sum_{j \in \mathcal{J}} f_{vu,ijk} = \sum_{v:(u,v) \in \mathcal{E}} \sum_{j \in \mathcal{J}} f_{uv,ijk} + x_{iuk}a_{ik}, i \in \mathcal{I}, u \in \mathcal{V}, k \in \mathcal{K}, u \neq i \quad (10)$$

Fractional Constraints (1)

- **Physical goods model**: facility capacity constraints $\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} x_{ijk} \leq b_j y_j$

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$$\sum_{i^* \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{i^* k} \frac{x_{i^* jk}}{x_{i^* jk} + \sum_{\ell \in \mathcal{L} \setminus \{i^*\}} x_{\ell jk}} \leq b_j y_j \quad (11)$$

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$$\sum_{i^*\in\mathcal{I}}\sum_{k\in\mathcal{K}} a_{i^*k}\frac{x_{i^*jk}}{x_{i^*jk} + \sum_{\ell\in\mathcal{L}\setminus\{i^*\}} x_{\ell jk}} \leq b_j y_j \qquad (11)$$

  - To linearize these constraints: first define a new variable ξ_{jk} such that

$$\xi_{jk} = \frac{1}{x_{i^*jk} + \sum_{\ell\in\mathcal{L}\setminus\{i^*\}} x_{\ell jk}} \qquad (12)$$

  - Condition equivalent to

$$\xi_{jk}\left(x_{i^*jk} + \sum_{\ell\in\mathcal{L}\setminus\{i^*\}} x_{\ell jk}\right) = \sum_{i^*\in\mathcal{L}} \xi_{jk}x_{i^*jk} = 1 \qquad (13)$$

  - In terms of ξ_{jk}, facility capacity constraints can then be rewritten as ($i^* \to i$)

$$\sum_{i\in\mathcal{I}}\sum_{k\in\mathcal{K}} a_{ik}\xi_{jk}x_{ijk} \leq b_j y_j \qquad (14)$$

$$\sum_{i\in\mathcal{L}} \xi_{jk}x_{ijk} = 1 \qquad (15)$$

Fractional Constraints (2)

- Theorem: polynomial mixed term $z = x.y$ ($x \triangleq$ binary variable, $y \triangleq$ continuous variable), can be represented by linear inequalities:
  1) $z \leq Ux$
  2) $z \leq y + L(x - 1)$
  3) $z \geq y + U(x - 1)$
  4) $z \geq Lx$

  where, U and L are upper and lower bounds of variable y, i.e., $L \leq y \leq U$

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  where, U and L are upper and lower bounds of variable y, i.e., $L \leq y \leq U$
- Introduce auxiliary variable $\zeta_{ijk} = \xi_{jk} x_{ijk}$, where $\xi_{jk} \triangleq$ fraction such that $L(=0) \leq \xi_{jk} \leq U(=1)$, to obtain:

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} \zeta_{ijk} \leq b_j y_j \tag{16}$$

$$\sum_{i \in \mathcal{L}} \zeta_{ijk} = 1 \tag{17}$$

$$\zeta_{ijk} \leq x_{ijk} \tag{18}$$

$$\zeta_{ijk} \leq \xi_{jk} \tag{19}$$

$$\zeta_{ijk} \geq \xi_{jk} - (1 - x_{ijk}) \tag{20}$$

$$\zeta_{ijk} \geq 0 \tag{21}$$

Fractional Constraints (3)

- **Linearization**
  - Increases complexity of the model: addition of $(I+1).J.K$ auxiliary variables ζ_{ijk} and ξ_{jk} together with $(4.I+1).J.K$ associated constraints
  - Works for small-size problems but gap between IP and LP relaxation may become huge for larger problems
  - Set $\mathcal{L}$ a priori unknown

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Facility capacity constraints

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} \frac{x_{ijk}}{\sum_{\ell \in \mathcal{L}} x_{\ell jk}} \leq b_j y_j, \forall j \in \mathcal{J}$$

- Explicit dependence on product index k in LHS of facility capacity constraints prevents per-product formulation
- Capacity sharing among K product types more complex structure than superposition of K independent facility capacity constraints

Approximation (1)

- Start from single product formulation: facility capacity constraints formulated for single-product model ($K = 1$):

$$\sum_{i \in \mathcal{I}} a_i \frac{x_{ij}}{\sum_{\ell \in \mathcal{L}} x_{\ell j}} \leq b_j y_j, j \in \mathcal{J}$$

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- Assume inequality verified for each k independently (dedicated capacity per-product type):

$$\sum_{i \in \mathcal{I}} a_{ik} x_{ijk} \leq b_{jk} \sum_{i \in \mathcal{L}} y_j x_{ijk}, j \in \mathcal{J}, k \in \mathcal{K}$$

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- Start from single product formulation: facility capacity constraints formulated for single-product model ($K = 1$):

$$\sum_{i \in \mathcal{I}} a_i \frac{x_{ij}}{\sum_{\ell \in \mathcal{L}} x_{\ell j}} \leq b_j y_j, j \in \mathcal{J}$$

- Move denominator out of LHS:

$$\sum_{i \in \mathcal{I}} a_i x_{ij} \leq b_j \sum_{i \in \mathcal{L}} y_j x_{ij}, j \in \mathcal{J}$$

- Assume inequality verified for each k independently (dedicated capacity per-product type):

$$\sum_{i \in \mathcal{I}} a_{ik} x_{ijk} \leq b_{jk} \sum_{i \in \mathcal{L}} y_j x_{ijk}, j \in \mathcal{J}, k \in \mathcal{K}$$

- Re-introduce summation over k (in both members):

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} x_{ijk} \leq \sum_{k \in \mathcal{K}} (b_{jk} \sum_{i \in \mathcal{L}} y_j x_{ijk}), j \in \mathcal{J}$$

Approximation (2)

- Transformation removes fractional term (LHS) but introduces sum over individual product capacity (b_{jk})
  $\Rightarrow$ No apparent gain from this transformation ?

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- **Assumption**: products homogeneously distributed among installed facilities
  $\rightarrow b_j = Kb_{jk}$ (remove dependence on per-product capacity distribution)

  $\Rightarrow$ Inequalities for facility capacity constraints (29) when $\mathcal{L} \rightarrow \mathcal{I}$: identical demands assigned to same facility j

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} x_{ijk} \leq \frac{1}{K} b_j y_j \sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} x_{ijk}, \forall j \in \mathcal{J} \tag{22}$$

  $\Rightarrow$ Inequalities for facility capacity constraints (29) when $|\mathcal{L}| \rightarrow 1$: each product type-size pair assigned to single demand

$$\sum_{i \in \mathcal{I}} \sum_{k \in \mathcal{K}} a_{ik} x_{ijk} \leq \frac{1}{K} b_j y_j \sum_{k \in \mathcal{K}} x_{*jk}, \forall j \in \mathcal{J} \tag{23}$$

Approximation (3)

- **Scenario**: Set of disjoint demands wrt product type k of same size s: pairs $(k_1, s), (k_2, s), \ldots, (k_K, s)$
  With $K = I$ pairs, i.e., one per customer $i \in \mathcal{I} = \mathcal{K}$: total capacity required $= K.s$
- If $b_j = s$ (and facility installation cost low enough to steer local assignment)
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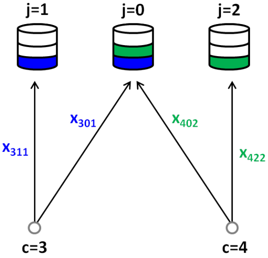

Case
x_{311} = a → 1
x_{301} = 1 – a → 0 (routing cost → 0)
x_{422} = b → 1
x_{402} = 1 – b → 0 (idem)
⇒ **1 unit on j=1 and j=2 + 1 unit on j = 3 (almost unused)**

With K products of same size s: over-dimensioning by factor K

Additional Constraints

Consider simplified objective:

$$\min \sum_{(u,v)\in\mathcal{E}} \tau_{uv} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} f_{uv,ijk} + \sum_{j\in\mathcal{J}} \mathtt{f}_j y_j \qquad (24)$$

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with additional constraints:

- Aggregated capacity constraints $\sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} a_{ik} \leq \frac{1}{K} \sum_j b_j y_j \sum_i \sum_k x_{ijk}$
- Individual fractions remain within $[0, 1]$, i.e., $0 \leq x_{ijk} \leq 1$
- At least one facility shall be opened $\sum_{j\in J} y_j \geq 1$
  Particular case: divide total demand size by per-facility capacity b_j such that min.number of facilities $\leq \sum_{j\in J} y_j$
- All product types covered by installed facilities $\sum_{j\in J} \sum_{k\in\mathcal{K}} z_{jk} \geq K$

MIP Formulation

$$\min \sum_{(u,v)\in\mathcal{E}} \tau_{uv} \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} f_{uv,ijk} + \sum_{j\in\mathcal{J}} \varphi_j y_j \quad (25)$$

subject to:

$$\sum_{j\in\mathcal{J}} x_{ijk} = 1 \qquad i\in\mathcal{I}, k\in\mathcal{K}, a_{ik}>0 \quad (26)$$

$$z_{jk} \le y_j \qquad j\in\mathcal{J}, k\in\mathcal{K} \quad (27)$$

$$x_{ijk} \le z_{jk} \qquad i\in\mathcal{I}, j\in\mathcal{J}, k\in\mathcal{K} \quad (28)$$

$$\sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} a_{ik} x_{ijk} \le \frac{1}{K} b_j y_j \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} x_{ijk} \qquad j\in\mathcal{J} \quad (29)$$

$$\sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} a_{ik} \le \frac{1}{K} \sum_{j\in\mathcal{J}} b_j y_j \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} x_{ijk} \quad (30)$$

$$f_{uv,ijk} \le \min(q_{uv}, a_{ik} x_{ijk}) \qquad (u,v)\in\mathcal{E}, i\in\mathcal{I}, j\in\mathcal{J}, k\in\mathcal{K} \quad (31)$$

$$\sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} f_{uv,ijk} \le q_{uv} \qquad (u,v)\in\mathcal{E} \quad (32)$$

$$a_{ik} x_{iik} + \sum_{v\in\mathcal{V}:(i,v)\in\mathcal{E}} \sum_{j\in\mathcal{J}} f_{iv,ijk} = a_{ik} \qquad i\in\mathcal{I}, k\in\mathcal{K}, i\ne j, a_{ik}>0 \quad (33)$$

$$\sum_{v:(v,u)\in\mathcal{E}} \sum_{j\in\mathcal{J}} f_{vu,ijk} = \sum_{v:(u,v)\in\mathcal{E}} \sum_{j\in\mathcal{J}} f_{uv,ijk} + a_{ik} x_{iuk} \qquad i\in\mathcal{I}, u\in\mathcal{V}, k\in\mathcal{K}, u\ne i \quad (34)$$

$$x_{ijk} \in [0,1] \qquad i\in\mathcal{I}, j\in\mathcal{J}, k\in\mathcal{K} \quad (35)$$

$$y_j \in \{0,1\} \qquad j\in\mathcal{J} \quad (36)$$

$$z_{jk} \in \{0,1\} \qquad j\in\mathcal{J}, k\in\mathcal{K} \quad (37)$$

$$f_{uv,ijk} \ge 0 \qquad (u,v)\in\mathcal{E}, i\in\mathcal{I}, j\in\mathcal{J}, k\in\mathcal{K} \quad (38)$$

Performance benchmark

Goals

- Computational performance evaluation (computational time and solution quality) using `CPLEX 12.6`
- Target computational time upper bound of 900s (average roll-out time)

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- Network topology of 25 nodes and 90 arcs
- Tuning facility capacity and associated costs

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Execution

- Barrier algorithm at root (`rootalg = 4`)
- Barrier algorithm at other nodes (`nodealg = 4`)
- Balance feasibility and optimality (`mipemphasis = 1`)

Performance benchamrk: results

Scenario	Root time (s)	Total time (s)	Final Gap (%)
sc-0k9-0k9	3037	3294	0.00
sc-1k-1k	2516	2773	0.00
sc-1k2-1k2	2306	2565	0.00
sc-1k5-1k5	2506	2763	0.00
sc-1k8-1k8	2921	3179	0.00
sc-2k-2k	3111	5360	0.00
sc-2k25-2k25	2912	5706	0.00
sc-3k-3k	2616	7189	0.00
sc-3k6-3k6	3270	5967	0.00
sc-4k5-4k5	3309	6029	0.00
sc-6k-6k	2895	9664	0.00
sc-9k-9k	3241	5493	0.00
Avg	2887	4999	0.00
Stdev	333	2164	0.00

Scenario	Root time (s)	Total time (s)	Final Gap (%)
sc-0k9-2k	3092	3353	0.00
sc-1k-2k	3206	3463	0.00
sc-1k2-2k	3282	3541	0.00
sc-1k5-2k	3337	3595	0.00
sc-1k8-2k	2824	3080	0.00
sc-2k-2k	3120	5405	0.00
sc-2k25-2k	2977	5468	0.00
sc-3k-2k	2847	9365	0.00
sc-3k6-2k	2759	5103	0.00
sc-4k5-2k	2880	5365	0.00
sc-6k-2k	3558	5490	0.00
sc-9k-2k	2628	4931	0.00
Avg	2808	4474	0.00
Stdev	272	1716	0.00

Evaluation instances: topologies and demands

- Topologies (SNDLib database)

Topology	Nodes	Links	Min,Max,Avg Degree	Diameter
abilene	12	15	1;4;2.50	3
atlanta	15	22	2;4;2.93	3
france	25	45	2;10;3.60	8
geant	22	36	2;8;3.27	5
germany50	50	88	2;5;3.52	9
india35	35	80	2;9;4.57	7
newyork	16	49	2;11;6.12	2
norway	27	51	2;6;3.78	7

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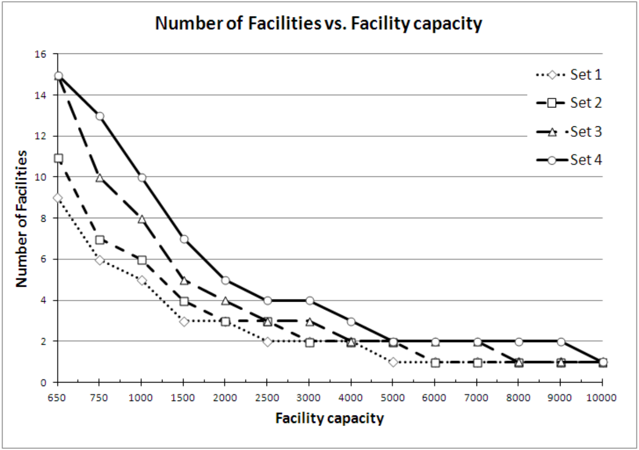
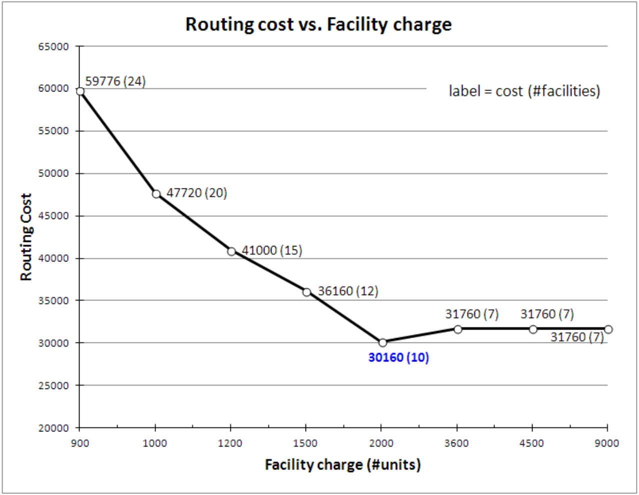
- Links capacity and cost from SNDlib database
- Demands
  - Produce set of ten problem instances with 3000 demands
  - Demands generated using following distributions:
    - **Demand size**: Pareto distribution commonly used to model file size $f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}, \ x \geq x_m$
    - **Demand frequence**: Generalized Zipf-Mandelbrot distribution (frequency of event occurrence inversely proportional to its rank)

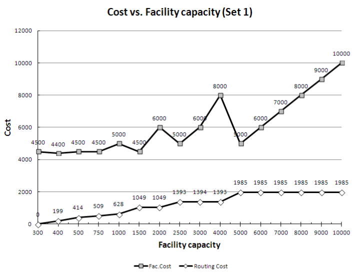
Results: Number of facilities vs. Facility Capacity

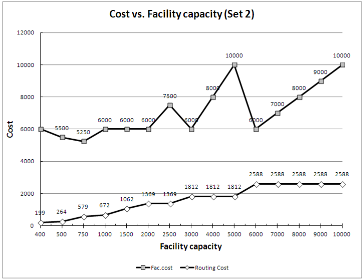


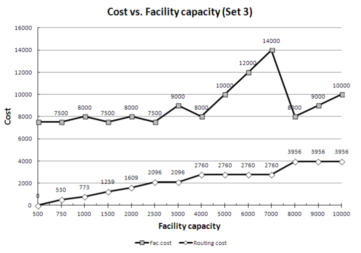
Results: Routing Cost vs. Facility Charge

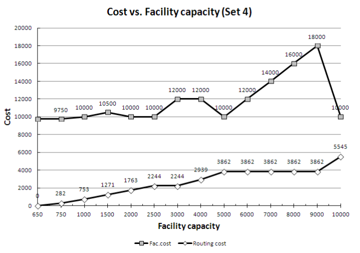


Deeper look (1): Digital goods model

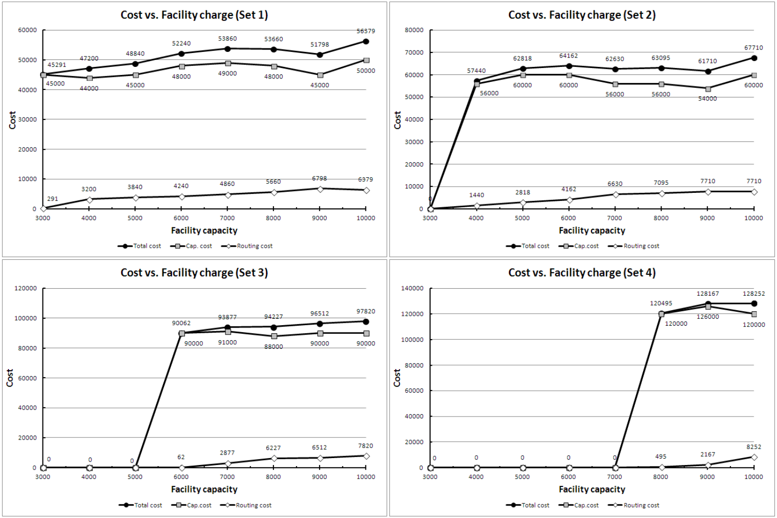








Deeper look (2): Physical goods model



- Reliability based on **levels assignments** strategy: r $(r = 0, \ldots, J-1)$ level at which a facility serves a given customer demand
  - r=0: primary assignment
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  - and so on

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$$\sum_{j \in \mathcal{J}} \varphi_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} \sum_{r=0}^{J-1} d_{ij} a_{ijk} x_{ijkr} q^r (1-q) \tag{39}$$

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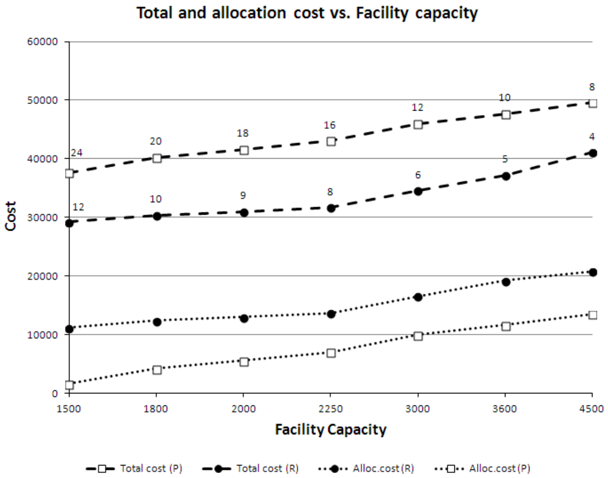
  - First term: total fixed installation cost
  - Second term: expected transport cost where facility j serves customer i demand if
    - its lower-level assigned facilities all disrupted: occurrence probability q^r
    - and facility j still available: occurrence probability $1q$

Results: Demand Protection (cRFLP) vs. Rerouting (MSMP-cFLFRP)



- As facility capacity increases, total cost (R) of re-routing strategy remains lower than total cost (P) of protection strategy (two levels of protection)
  - Higher allocation cost required by cRFLP compared to MSMP-cFLFRP because of smaller number of installed facilities
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Results: Main observations

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- Implication: routing metric would require accounting from facility load distribution and data availability

Conclusion and Future Work

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- Propose a mixed-integer formulation for combined multi-source multi-product capacitated facility location-flow routing problem (MSMP-cFLFRP)
- Our formulation accounts for specifics of digital object storage and supply
  Note: known formulations translate multi-product problem as single-commodity problem solved separately for each product
- Approximation of fractional constraints enables to solve to optimality small- to medium-size instances with an order of thousands of demands
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Future Work

- Improve computation method to avoid excessive computation time on large-instances with order of 10k demands