The k-Node-Connected Subgraph Problem: Valid inequalities and Branch-and-Cut

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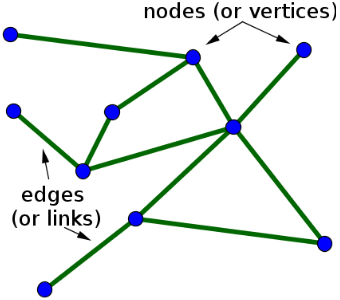
Outline

1. The kNCSP
   - Formulation
   - Valid inequalities
2. Structural properties and reduction operations
3. Branch-and-Cut
4. Conclusion and Perspectives

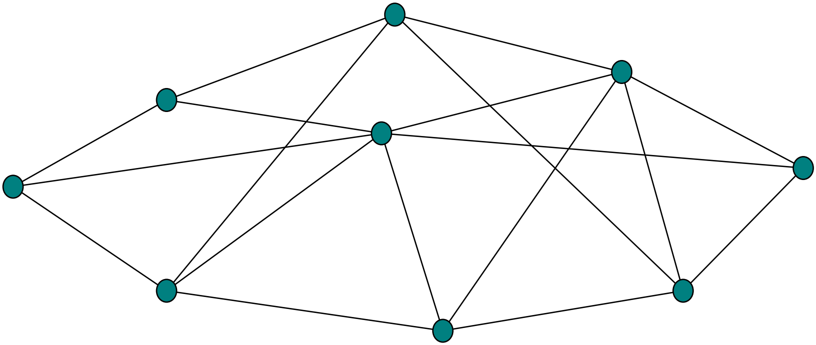
The kNCSP

Definition

A graph $G = (V, E)$ is called k-node-connected if for every pair of nodes $i, j \in V$, there are at least k node-disjoint paths between i and j.



The kNCSP



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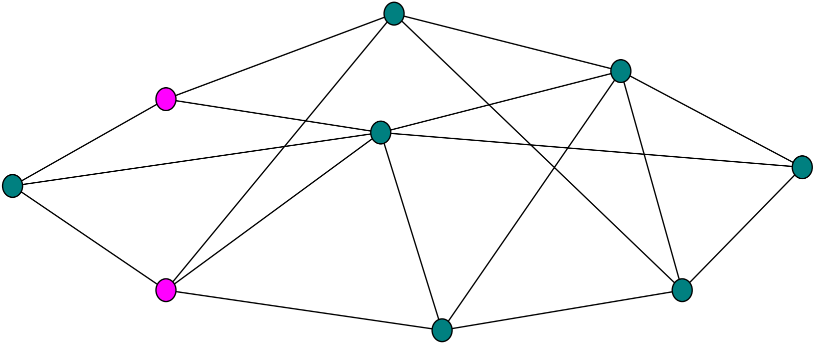
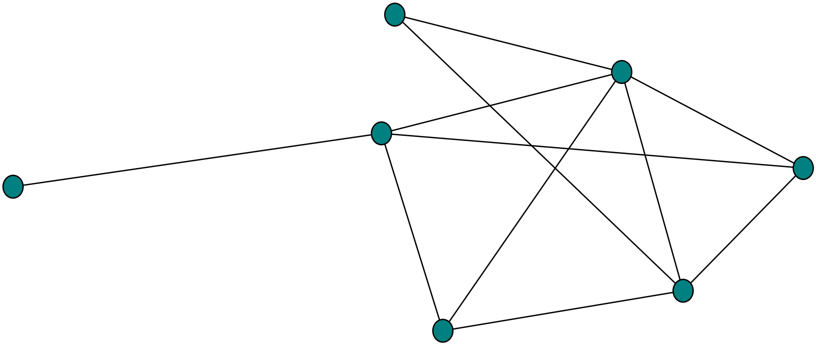


The *k*NCSP



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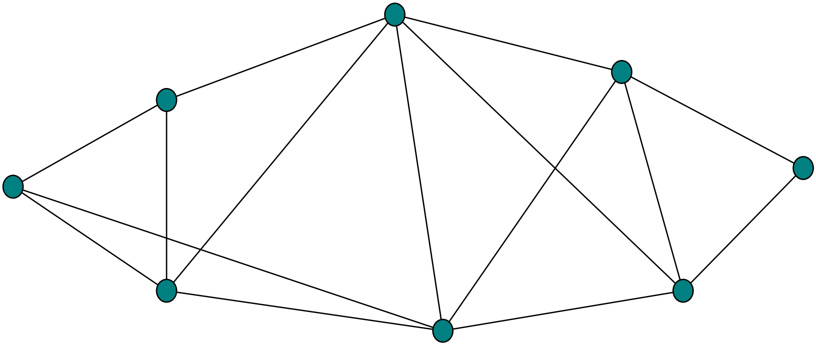

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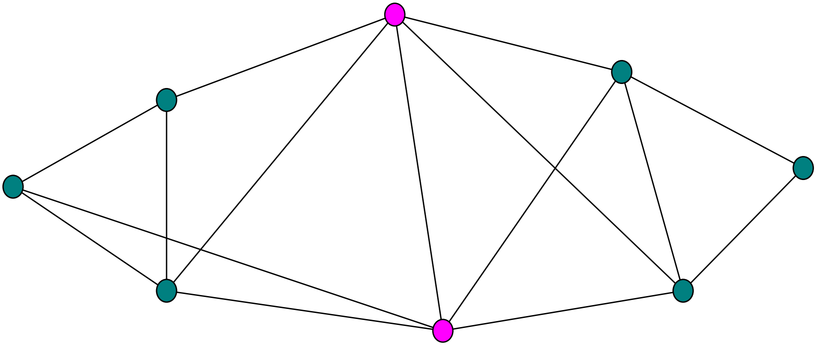

The *k*NCSP

The kNCSP

Given a weight function c on E that associates with an edge $e \in E$ the weight $c(e) \in \mathbb{R}$, the *k-Node-Connected Subgraph Problem* is to find a k-node-connected spanning subgraph $H = (V, F)$ of G such that $\sum_{e \in F} c(e)$ is minimum.

Related work

- The kNCSP is NP-hard for $k \geq 2$ [Garey and Johnson (1979)].

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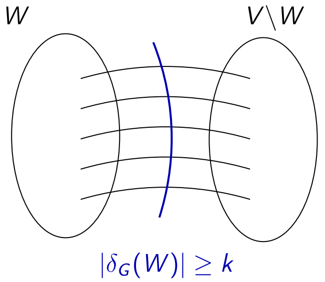
- Grötschel, C.L. Monma and M. Stoer (1991) studied k-node-connected graphs within a more general survivability model.

Formulation

Cut inequalities

Every cut must contain at least k edges.

$$x(\delta_G(W)) \geq k$$

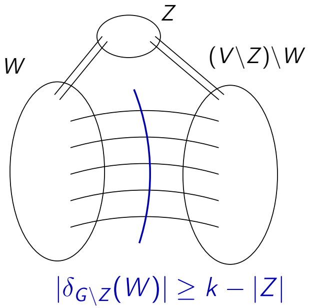


Formulation

Node-cut inequalities

Every node-cut must contain at least $k - |Z|$ edges.

$$x(\delta_{G\setminus Z}(W)) \geq k - |Z|$$



Integer programming formulation of kNCSP

$$\text{Minimize} \sum_{e \in E} c(e)x(e)$$

s.t.

$$x(e) \geq 0 \qquad e \in E, \tag{1}$$

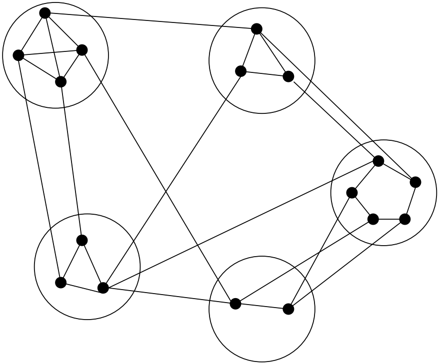
$$x(e) \leq 1 \qquad e \in E, \tag{2}$$

$$x(\delta_G(W)) \geq k \qquad \emptyset \neq W \subseteq V, \tag{3}$$

$$x(\delta_{G \setminus Z}(W)) \geq k - |Z| \qquad \begin{array}{l} \emptyset \neq Z \subseteq V;\ |Z| \leq k-1, \\ \emptyset \neq W \subseteq V \setminus Z. \end{array} \tag{4}$$

Notations

Given a partition $\pi = (V_1, ..., V_p)$, $p \geq 2$, we denote by G_π the subgraph induced by π, that is, the graph obtained by contracting the sets V_i, $i = 1, ...p$.
We denote by $\delta_G(\pi)$ the edge set of G_π.



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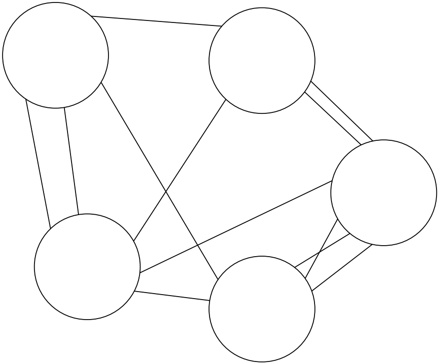


Node-partition inequalities

Theorem

Consider a subset $Z \subset V$, such that $|Z| \leq k-1$, and let $\pi = V_1, ..., V_p$, $p \geq 2$ be a partition of $V \backslash Z$. The following inequality is valid for kNCSP(G).

$$x(\delta_{G \backslash Z}(\pi)) \geq \begin{cases} \lceil \frac{p(k-|Z|)}{2} \rceil & \textit{if} \quad |Z| \leq k-2 \\ p-1 & \textit{if} \quad |Z| = k-1. \end{cases} \tag{5}$$

SP-node-partition inequalities

Theorem

Consider a partition $\pi = (V_1, ..., V_p)$ of $V \backslash Z$, such that G_π is series-parallel, then the inequality

$$x(\delta_{G \backslash Z}(\pi)) \geq \lceil \frac{k - |Z|}{2} \rceil p - 1 \qquad (6)$$

is valid for kNCSP(G).

SP-node-partition inequalities

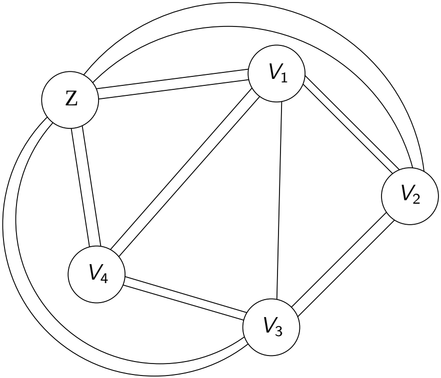


Figure : An SP-node-partition configuration with $k = 5$ and $Z = 2$

SP-node-partition inequalities

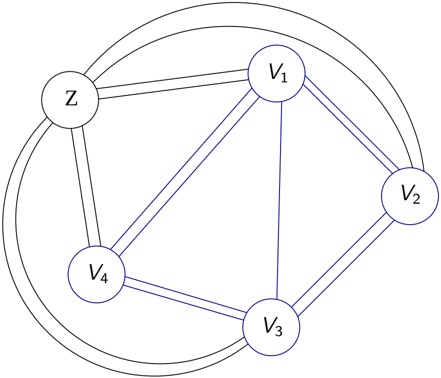


Figure : An SP-node-partition configuration with $k = 5$ and $Z = 2$

F-node-partition inequalities

Theorem

Let $Z \subset V$ with $|Z| \leq k-1$. Consider a partition $\pi = (V_0, ..., V_p)$ of $V \backslash Z$, and $Z_i = \{z \in Z \mid \exists\, e \in \delta(\{z\}, V_i)\}$ for $i = 1, ..., p$, and let F be a subset of $\delta_{G\backslash Z}(V_0)$ such that $\sum_{i=0}^{p}(k - |Z_i|) - |F|$ is odd. Then the inequality

$$x(\delta_{G\backslash Z}(\pi \backslash F)) \geq \lceil \frac{\sum_{i=0}^{p}(k - |Z_i|) - |F|}{2} \rceil \quad (7)$$

is valid for kNCSP(G).

Facets of kNCSP(G)

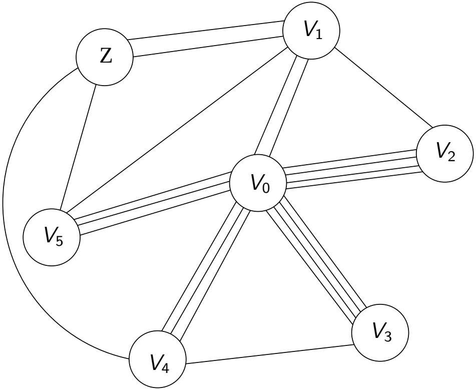


Figure : An F-node-partition configuration with $k = 5$

Structural properties

Let $\overline{x}$ be an extreme point of $P(G,k)$.

$$
(Q)\begin{cases}
x(e) = 1, & \forall\ e \text{ such that } \quad \overline{x}(e) = 1,\\
x(e) = 0, & \forall\ e \text{ such that } \quad \overline{x}(e) = 0,\\
x(\delta_G(W)) = k, & \forall \quad \delta_G(W) \in \mathscr{C}_e^*(\overline{x}),\\
x(\delta_{G\setminus Z}(W)) = k - |Z|, & \forall \quad \delta_{G\setminus Z}(W) \in \mathscr{C}_n^*(\overline{x}),
\end{cases}
$$

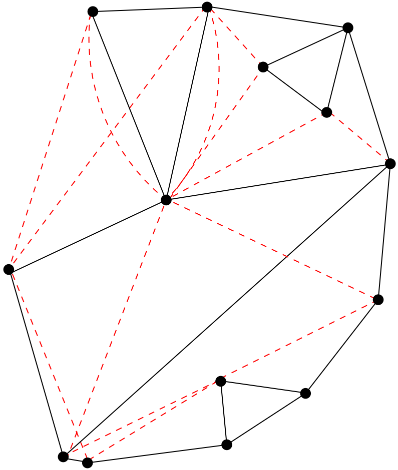
Proposition

Let $\delta(W)$ be a cut of G tight for $\overline{x}$. Then system (Q) can be chosen so that $\mathscr{C}_e^*(\overline{x}) \cup \mathscr{C}_n^*(\overline{x}) \subseteq \mathscr{C}(\overline{x}, W)$.

Reduction operations

- θ_1 : Delete an edge $e \in E$ such that $\overline{x}(e) = 0$.
- θ_2 : Contract a node subset $W \subseteq V$ such that $G[W]$ is k-edge connected, $\overline{x}(e) = 1$ for all $e \in E(W)$ and $\overline{x}(\delta(W)) = k$.
- θ_3 : Contract a node subset $W \subseteq V$ such that $|W| \geq 2$, $|\overline{W}| \geq 2$, $\overline{x}(e) = 1$ for all $e \in E(W)$, and $|\delta_G(W)| = k$.
- θ_4 : Replace a set of parallel edges by only one edge.

Reduction operations



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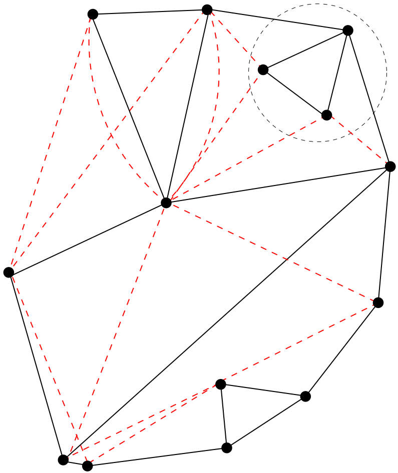

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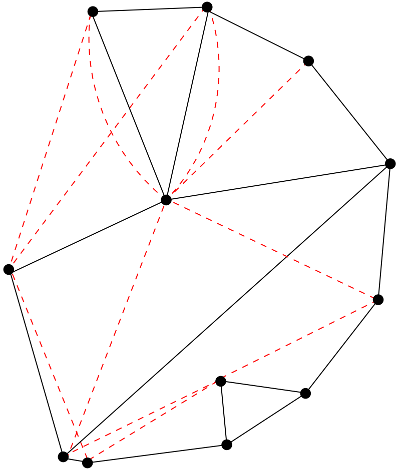

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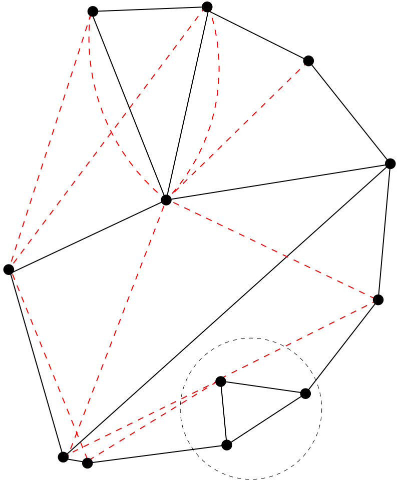

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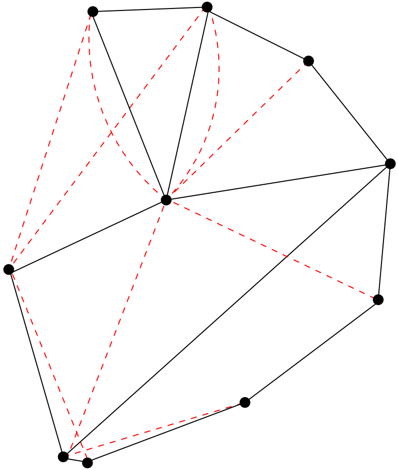

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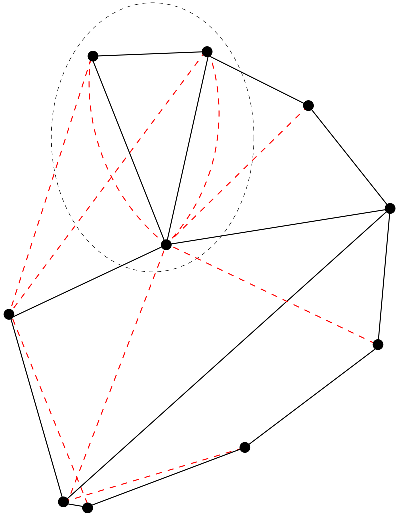

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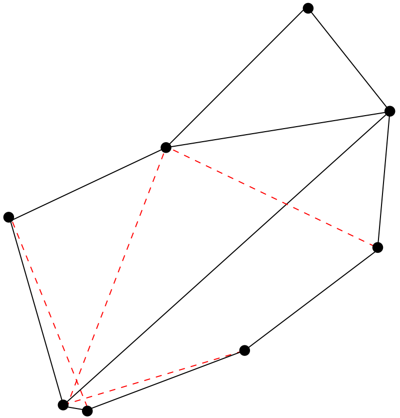


Reduction operations

Proposition

Let $G' = (V', E')$ and $\overline{x}'$ be the graph and the solution obtained from G and $\overline{x}$, respectively, by the application of Operations $\theta_1, ..., \theta_4$. Then $\overline{x}'$ is an extreme point of $P(G', k)$.

Branch-and-Cut

- The algorithm has been implemented in C++ using CPLEX 12.5 with the default settings.
- All experiments were run on a 2.10GHzx4 Intel Core(TM) i7-4600U running linux with 16 GB of RAM.
- We have tested our approach on several instances derived from SNDlib and TSPlib topologies.

Branch-and-Cut

Instance	k	#EC	#NC	#FNPC	#SPNC	#NPC	COpt	Gap(%)	NSub	CPU
polska_12	3	7	121	0	5	0	51	0.00	1	0 :00 :04
france_25	3	71	807	8	14	1	3254	0.39	17	1 :03 :55
india_35	3	18	270	9	3	0	489	1.45	2	2 :04 :21
pioro_40	3	11	546	0	2	1	5637	0.00	1	0 :42 :07
berlin_52	3	95	914	14	83	0	16524	0.09	6	3 :14 :23
eil_76	3	85	1674	9	142	1	-	*0.12*	6	5 :00 :00
atlanta_15	4	0	72	2	-	0	4615	0.00	1	0 :00 :25
sun_27	4	0	9172	74	-	0	6867	0.00	1	0 :21 :31
india_35	4	25	4165	7	-	2	547	6.45	2	2 :01 :59
pioro_40	4	18	598	4	-	0	8096	0.00	1	0 :31 :14
berlin_52	4	145	1045	19	-	3	18268	0.05	3	3 :14 :23
eil_76	4	34	1832	8	-	0	971	0.07	2	0 :24 :32
france_25	5	0	21284	0	2	6	6439	0.91	5	2 :23 :00
bays_29	5	2	40972	0	0	1	28411	1.3	2	2 :12 :12
india_35	5	0	41344	0	4	2	638	2.91	4	1 :46 :04
pioro_40	5	8	1342	7	3	0	11756	0.25	4	0 :58 :24
berlin_52	5	76	3451	25	0	5	21763	0.15	5	4 :14 :23
st_70	5	4	847	21	1	0	-	9.12	1	5 :00 :00

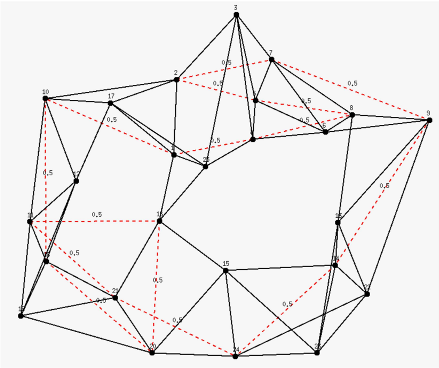
Table : Results for $k = 3, 4, 5$

Branch-and-Cut

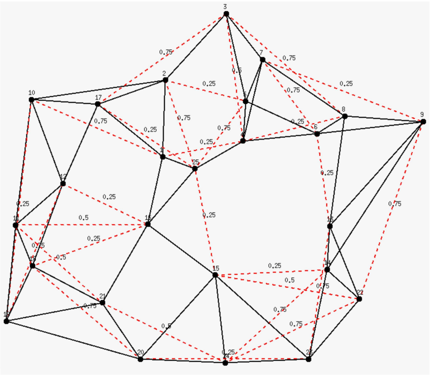
Instance	#EC	#NC	#FNPC	#SPC	#NPC	COpt	Gap(%)	NSub	CPU
abilene_12	20	2211	0	3	2	214	1.29	12	0 :00 :18
nobel-us_14	29	775	4	7	7	219	0.57	44	0 :01 :15
atlanta_15	16	143	6	1	1	3265	0.15	3	0 :01 :03
nobel_germany_17	83	5607	7	4	1	53	1.3	21	0 :14 :52
ulysses22_22	49	22633	5	0	5	141	0.74	10	0 :21 :08
janos-us_26	42	35782	6	2	0	282	1.45	26	3 :14 :53
sun_27	41	910	9	8	0	4771	0.87	6	3 :31 :11
norway_27	48	1214	5	7	6	6864	2.32	2	2 :02 :07
bays_29	66	227	2	8	6	14791	3.1	6	2 :03 :52
india_35	18	270	9	3	0	489	1.45	2	2 :04 :21
pioro_40	33	725	1	5	2	-	7.25	14	5 :00 :00

Table : Results for $k = 3$ without reduction operations

Branch-and-Cut



With the additional valid inequalities



Without the additional valid inequalities

Conclusion

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Conclusion

- The k-node connected subgraph problem.
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- Branch-and-Cut algorithm.

Perspectives

- Investigate the structural properties of the linear relaxation.

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- Investigate the structural properties of the linear relaxation.
- Study the problem when a bound is considered on the connectivity paths.

References

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Thank you for your attention